Funding Constraints and Informational Efficiency *

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Abstract

We develop a tractable rational expectations model that allow for general portfolio constraints. We apply our methodology to study a model where constraints arise due to endogenous margin requirements. We argue that margin requirements affect and are affected by informational efficiency, leading to a novel amplification mechanism. A drop in investors' wealth tightens constraints and reduces their incentive to acquire information, which lowers price informativeness. Moreover, financiers who use information in prices to assess the risk of financing a trade face more uncertainty and set higher margins, which further tightens constraints. This *information spiral* implies that risk premium, conditional volatility and Sharpe ratios rise disproportionately as investors' wealth drops. Our model uncovers a new, information-based rationale why the initial wealth of investors is important.

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1 Introduction

One of the basic tenets of financial economics is that market prices aggregate information of investors. The core of the argument is that investors acquire information about future asset values and trade on it, thereby impounding that information into price. This argument relies on investors' incentives to acquire information and their capacity to trade on it, both of which are crucially affected by their ability to fund their trades. This raises an important question: how do funding constraints faced by investors affect price informativeness? Moreover, since the information in prices can be useful for financiers to assess the risk of financing a trade, another important question is how price informativeness affects the tightness of funding constraints. To answer these questions, one needs a model in which price informativeness and funding constraints are both jointly determined in equilibrium. In this paper, we present such a model and study its asset pricing implications.

The main challenge in studying the interplay between funding constraints and informational efficiency is that most of the existing noisy rational expectation equilibrium (REE) models, which are instrumental for analyzing informational efficiency, could not accommodate constraints in a tractable manner.[1] Our first contribution is to develop a tractable REE model with *general portfolio constraints* that can depend on prices. We apply our methodology to study a model where portfolio constraints arise due to endogenous margin requirements. Our second contribution is to show that the funding of investors affects and is affected by informational efficiency, leading to a novel amplification mechanism, which we call *information spiral.* This mechanism implies that risk premium, conditional volatility and Sharpe ratios rise disproportionately as investors' wealth drops.

We consider a canonical CARA-normal REE model in which investors trade to profit from their private signals about the risky asset's fundamental value and to hedge their endowment shocks. The novelty is that we allow for general portfolio constraints: investors can

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[1]Two important exceptions are Yuan (2005) and Nezafat, Schroder, and Wang (2017), which only analyze a special type of borrowing constraint and short-sale constraint, respectively.

only trade up to some maximal long and short positions of the risky asset, and these portfolio constraints can be any functions of price. This general, price-dependent specification of portfolio constraints nests many types of real-world trading constraints such as short-sale constraints, borrowing constraints, margin requirement, etc. Without the constraints, the model is standard: (1) an investor's demand is linear in his private signal, the endowment shock, and the price, (2) the equilibrium price itself is linear in the fundamental value and aggregate endowment shock, and (3) investors' initial wealth is irrelevant for asset prices.

Under portfolio constraints, the equilibrium is as follows: (1) investors' *desired* demand, i.e. the amount they would like to trade, is still linear whereas their *actual* demand is the desired demand truncated to the maximal long or short positions, (2) although the price function is potentially non-linear, it is informationally equivalent to a linear combination of the fundamental value and the aggregate endowment shock, so that the inference is still tractable, and (3) investors' initial wealth matters for asset prices if it affects constraints.

We then apply our methodology to study how portfolio constraints affect informational efficiency. We show that constraints harm informational efficiency via an *information production channel*.[2] Intuitively, when constraints become tighter, investors can only take smaller positions hence profit less on their private information. In anticipation, they would acquire less information. As all investors ex-ante acquire less information, price becomes less informative about asset fundamentals in equilibrium. We obtain these results using the simple expression we obtain for the marginal value of information of an investor who faces general portfolio constraints.[3] This expression can be useful in a broad class of applications in which investors face portfolio constraints and acquire information.

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[2]One might expect that the portfolio constraints per se could reduce informational efficiency because demands of constrained informed investors cannot respond to their private information. We show that this intuition is not correct because even in the constrained setting, the aggregate demand from investors continues to vary with fundamentals via the changes in the fraction of investors being constrained at the maximal long and short positions. For example, when the private information about the asset are more favorable, there will be more (fewer) investors being constrained at the maximal long (short) positions.

[3]The expression says that the ratio of marginal values of information for a constrained and unconstrained investor is equal to the ratio of utility a constrained investor gets in the states when his constraints do not bind to his total expected utility.

Next, we study the reverse channel on how informational efficiency affects funding constraints. Motivated by real-world margin constraints as argued in Brunnermeier and Pedersen (2009), we assume investors finance their positions through collateralized borrowing from financiers who require the margins to control their value-at-risk(VaR). We further assume that financiers use information in prices when setting the margin requirement. We argue that lower informational efficiency leads to tighter margins. The intuition is that, when prices are less informative, financiers who use information in prices to assess the risk of financing a trade face more uncertainty about fundamentals and thus set higher margins.

Our model implies that funding constraints affect and are affected by informational efficiency. In light of this, both margins and asset prices are determined jointly in equilibrium: investors and financiers determine demands and margins anticipating a particular price function and, in equilibrium, demands and margins are consistent with the anticipated one. We get our main result, a novel *information spiral* showed in Figure 1. With a negative wealth shock, constraints tighten, investors acquire less information, leading to lower informational efficiency in equilibrium. As price becomes less informative of the fundamentals, financiers tighten their margins requirement to satisfy their VaR constraints, further tightening investors' funding constraints. As a result, a small shock to wealth may have a profound effect on information production, informational efficiency and funding constraints.

Our information spiral suggests a novel amplifying mechanism on asset prices. We show that a small shock to investors' wealth can lead to large increase in conditional volatility, risk premium and sharpe ratio of the asset. Each of these results match empirical observations during crises.[4] While the literature has proposed other amplifying mechanisms for the effect of wealth shocks, ours is unique in the sense that it acts through informativeness of the financial markets, which could have further macro-economic consequences given the central role of the stock market in the real economy.

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[4]Financial crises, such as the hdge fund crisis of 1998 or 2007/2008 subprime crisis, have several common characteristics: risk premia rise, conditional volatility of asset prices rise and sharpe ratio rises.

Figure 1: Amplification loop
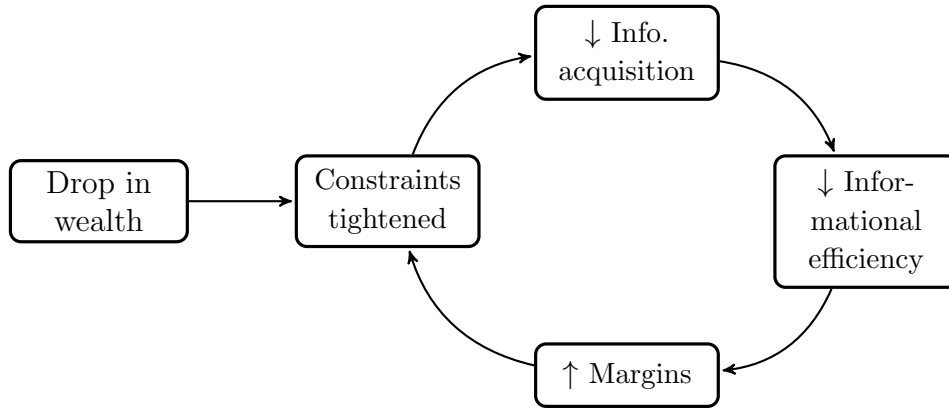
1.1 Related Literature

This paper lies at the intersection of various strands of literature. On the one hand, we share the emphasis of the work that studies the role played by financial markets in aggregating and disseminating information, following Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980a) and Diamond and Verrecchia (1981). In most of these models, it is generally assumed that investors can borrow freely at the riskless rate i.e., no funding constraints. From the methodological perspective, we contribute to this literature by developing a REE model that can incorporate general portfolio constraints. Similar to us, Yuan (2005) studies REE model with linear price dependent constraints. Our model also nests the model of Nezafat et al. (2017) which studies how short sale constraints affect information acquisition and asset prices. More recently, Albagli, Hellwig, and Tsyvinski (2011) develop a model that accommodates specific portfolio constraints (-1,+1) for risk-neutral investors. However, their model cannot accomodate general portfolio constraints (with risk-averse investor) which is the focus of our model.

Our work is related to the literature on information acquisition in REE models. Peng and Xiong (2006); Van Nieuwerburgh and Veldkamp (2009); Van Nieuwerburgh and Veldkamp (2010) study financial investor's information acquisition problem without funding constraints.

On the contrary, we study information acquisition incentives with funding constraints. We show that funding constraints affects and are affected by informational efficiency (through information acquisition of investors) which leads to an emergence of information spiral. Our paper also relates to the recent literature on the role of secondary financial markets as a primary source of information for decision makers. See Bond, Edmans, and Goldstein (2012) for recent survey on this topic. Goldstein, Ozdenoren, and Yuan (2013) show that the feedback effect from asset prices to the real value of a firm because capital providers learn from prices, generating complementarities in investments. Dow, Goldstein, and Guembel (2017) show that the feedback effect generates complementarities in the decision to produce information, but not in the trading decision. We contribute to this literature to study how financiers of investors can use the information in prices to set the margins requirement and we find that lower informational efficiency leads to tighter margins.

Finally, our work contributes to the literature on the effect of investors' wealth and the associated amplification mechanisms. For example, Xiong (2001) studies wealth constraint as an amplification mechanism, while Kyle and Xiong (2001) study it as a spillover mechanism. Gromb and Vayanos (2002, 2017) develop an equilibrium model of arbitrage trading with margin constraints to explain contagion. Brunnermeier and Pedersen (2009) studies how funding liquidity and market liquidity reinforce each other. He and Krishnamurthy (2011), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014) study how a fall in intermediary capital reduces their risk-bearing capacity and lead to rises in risk premium and conditional volatility. Overall, this literature emphasizes that leverage and asset prices need to be jointly determined (see review article by Fostel and Geanakoplos (2014)). Our paper is complementary to these studies. We show that informational efficiency amplifies the wealth effect in a REE model with a model of endogenous margin requirements. Our mechanism is novel in the sense that it involves changes in stock-market informativeness, which should be an important channel given the central role of the stock market in the real economy.

The reminder of the article is organized as follows. In section 2, we solve for the financial

market equilibrium and value of information in an REE model with general portfolio constraints. In section 3, we introduce margin requirements and argue that funding constraints affect and are affected by informational efficiency. In section 4, we explore the implications of this spiral for asset prices. Section 5 concludes.

2   An REE model with general portfolio constraints

In this section we introduce and solve the model with general portfolio constraints. In section 3.2, we endogenize the constraints and solve for a full equilibrium of the model.

2.1   Setup

There are three dates (i.e., $t \in \{0, 1, 2\}$) and two assets. The risk-free asset is the numeriaire. The payoff (fundamental value) of the risky asset is $f = v + \theta$ (which is paid at date 2), where v is the learnable component of fundamentals, $v \sim N(0, \tau_v^{-1})$ and θ is the unlearnable component of fundamentals, $\theta \sim \mathcal{N}\left(0, \tau_\theta^{-1}\right)$, and the aggregate supply of the asset is assumed to be constant 1 unit. The economy is populated by a unit continuum of investors, indexed by $i \in [0, 1]$, with identical risk-averse preferences over terminal wealth with constant absolute risk aversion parameter γ. Investors acquire information at date 0, learn endowment shocks and trade the risky asset at $t = 1$, and consume their assets' payoffs at $t = 2$.

At date 2, each investor receives an endowment b_i which includes revenues from other assets than stocks. The endowment payoff is independent of v, but it can be correlated with the unlearnable component of the risky asset, θ. We assume that the endowment is given by $b_i = e_i \theta + \eta_i$, where η_i is independent of θ and v. The coefficient e_i gives the sensitivity of the endowment shock to the unlearnable payoff of the risky asset. We also assume that investor knows his sensitivity e_i at the trading date ($t = 1$). Hereafter, we will refer to e_i as the endowment shock of investor i. Finally, the investor i's endowment e_i has systematic

and idiosyncratic components: $e_i = z + u_i$. All investors have beliefs that $z \sim \mathcal{N}(0, \tau_z^{-1})$ and $u_i \sim N(0, \tau_u^{-1})$, independent across investors and independent of z. This formulation implies that there is uncertainty about the aggregate endowment shocks z and this serves as noises in prices. Differences in exposures "e_i" across investors motivate trade in the risky asset.

At date 1, each investor i receives a signal $s_i = v + \epsilon_i$, where ϵ_i are i.i.d. with $\epsilon_i \sim N(0, \tau_{\epsilon_i}^{-1})$. The precision of her private signal τ_{ϵ_i} is optimally chosen by investor i at date 0, subject to a cost function $C(.)$. We assume that the cost function is identical to all investors and possesses standard characteristics: $C(\cdot)$ is continuous, $C(0) = C'(0) = 0$, and $C', C'' > 0$ for all τ_{ϵ_i}. When forming their expectations about the fundamental, investors use all the information available to them. The information set of investor i at time 1 is $\mathcal{F}_i = \{s_i, p, e_i\}$, where p is the equilibrium price at time 1. There is also a competitive market maker who has neither endowment shocks nor private information about the asset payoff. Hence, the market maker's information set at time 1 is $\mathcal{F}_m = \{p\}$ and she takes prices as given and submits her demand for the risky asset.

Constraints: Investors, but not the market maker, are subject to the following funding constraints: given the price p, the minimum and maximum positions that an investor can take are, respectively, $a(p)$ and $b(p)$. The functions $a(p)$ and $b(p)$ may depend on investors' initial wealth W_0 and the aggregate equilibrium parameters, such as price volatility. Where it does not cause confusion, we will not indicate this dependence explicitly. To summarize, at date 1 investors solve the following problem

$$\max_{x_i(p; s_i, e_i) \in [a(p), b(p)]} E[-\exp(-\gamma W_i) | s_i, e_i, p]$$

$$\text{subject to: } W_i = W_0 + x_i(v + \theta - p) + e_i \theta + \eta_i. \tag{1}$$

In the constraint above, W_0 denotes investors' initial wealth, the second term represents gain/loss from trading asset at time 1 and third and fourth terms denote the endowment shocks.

The market maker solves

$$\max_{x_m(p)} E[-\exp(-\gamma_m W_m)|p]$$

$$\text{subject to: } W_m = W_{0,m} + x_m(v + \theta - p). \tag{2}$$

where $\gamma_m \geq 0$ is her risk aversion. Finally, the equilibrium price is set to clear the market:

$$\int x_i(p; s_i, e_i)di + x_m(p) = 1.$$

2.2 Financial Market Equilibrium

We first solve for equilibrium in the unconstrained setting, which has already been studied in Biais, Bossaerts, and Spatt (2010) and Ganguli and Yang (2009). This is an important step in characterizing the equilibrium with constraints, which is why we review it here.

2.2.1 Unconstrained setting

We characterize the unconstrained equilibrium with its key features in the proposition below. Unless stated otherwise, proofs of all propositions are in the Appendix.

Proposition 1. *(Financial market equilibrium without portfolio constraints) Suppose investors have identical signal precision τ_ϵ and $\tau_u^2 \tau_\theta^2 < 3\gamma^2 (\tau_u + \tau_z)(\tau_\epsilon + \tau_v)$. Then there exists a unique equilibrium in which the price is informationally equivalent to a sufficient statistic $\phi^u = v - \frac{z}{\beta^u}$, which can be computed from price as follows: $\phi^u = f_0^u + f_1^u p$. The aggregate demand of investors and the market maker can be written as*

$$X^u(p, \phi) = c_0 + c_\phi \phi - c_p p, \qquad x_m^u = c_0^m + c_\phi^m \phi - c_p^m p.$$

The individual demand of investor i can be written as follows:

$$x_i^u = X^u + \xi_i, \ \xi_i \sim \mathcal{N}(0, \sigma_\xi^2).$$

And β_u is the unique root (β) which solves

$$\beta^3 \gamma (\tau_u + \tau_z) - \beta^2 \tau_u \tau_\theta + \beta \gamma (\tau_\epsilon + \tau_v) - \tau_\theta \tau_\epsilon = 0 \tag{3}$$

and β increases in τ_ϵ, the precision of information of investors.

All the coefficients are reported in the Appendix.

The analysis of unconstrained equilibrium highlights some important economics of the model which will continue to hold in the constrained setting. First, in equilibrium, price is informationally equivalent to a linear combination of the (learnable) fundamental payoff v and the aggregate endowment shock z. As we will show later, the linearity of the sufficient statistic is crucial in keeping our analysis of the constrained setting tractable. Second, how much information about the fundamental contained in price is captured by an endogenous, signal-to-noise variable β^u. More precisely, the conditional variance of the learnable fundamental decreases in β^u

$$\mathbb{V}(v|p) = \mathbb{V}(v|\phi^u) = (\tau_v + (\beta^u)^2 \tau_z)^{-1}$$

For this reason, we call β^u the *informational efficiency* of the market when investors are unconstrained in their trading. Importantly, the characterization of β^u in equation 3 shows that investors' information acquisition (higher signal precision τ_ϵ) improves the informational efficiency of the market.

2.2.2 Constrained setting

We now impose the portfolio constraints $a(p)$ and $b(p)$ into investors' problem. We guess and later verify, that there exists a *generalized linear equilibrium* in the economy, which we define as follows.

Definition 1. *An equilibrium is called generalized linear if there exists a function $f(p)$ and scalar β, such that $\phi = v - \frac{z}{\beta}$ is informationally equivalent to price and is given by $\phi = f(p)$.[5]*

The ϕ and β defined above are the obvious counterparts of ϕ^u and β^u in the economy without portfolio constraints. In a generalized linear equilibrium, despite the potential non-linearity of the price function, the sufficient statistic ϕ is still linear in (v, z) and thus normally distributed. Therefore, the inference from price remains tractable.

Now we proceed with the characterization of equilibrium in the constrained economy. We first define a function $T(x; a, b)$ that truncates its argument x to the interval $[a, b]$:

$$
T(x; a, b) = \begin{cases} x, & \text{if } a \leq x \leq b, \\ b, & \text{if } x > b, \\ a, & \text{if } x < a. \end{cases} \tag{4}
$$

Then we conjecture that the demand of an investor i in the constrained economy to be a truncation of his demand in the unconstrained economy x_i^u:

$$
\begin{aligned}
x_i &= T\left(x_i^u; a(p), b(p)\right) \\
&= T\left(X^u + \xi_i; a(p), b(p)\right)
\end{aligned} \tag{5}
$$

where X^u is the aggregate demand of investors in the unconstrained economy. We write the market maker's demand in the constrained economy as $x_m(p)$. Together, the market clearing

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[5]Our notion of generalized linear equilibrium follows Breon-Drish (2015).

condition can be written as

$$\int x_i di + x_m(p) = 1 \implies \int T\left(X^u + \xi_i; a(p), b(p)\right) di + x_m(p) = 1$$

Even though the market clearing condition looks intimidating, one can find a generalized linear equilibrium in this setup. The following proposition characterizes the equilibrium.

Proposition 2. *(Financial market equilibrium with portfolio constraints) Suppose investors face portfolio constraints and have identical signal variances τ_ϵ^{-1}. Then there is an equilibrium with informational efficiency β^c. In particular, there is a unique function $f(\cdot)$ such that*

$$f(p) = \phi \equiv v - (\beta^c)^{-1} z.$$

The function $f(p)$ satisfies the ODE

$$f'(p) = \frac{c_p^m + \pi_2 c_p - \pi_1 a'(p) - \pi_3 b'(p)}{c_\phi^m + \pi_2 c_\phi} \tag{6}$$

subject to the the boundary condition $f(0) = f_0$, where $\pi_1(p, \phi) = \Phi\left(\frac{a(p) - X^u(p,\phi)}{\sigma_\xi}\right)$ is the fraction of investors whose lower constraint binds, $\pi_3(p, \phi) = 1 - \Phi\left(\frac{b(p) - X^u(p,\phi)}{\sigma_\xi}\right)$ is the fraction of investors whose upper constraint binds, and $\pi_2 = 1 - \pi_1 - \pi_3$ is the fraction of unconstrained investors. The constant f_0 is the unique solution to

$$X(0, f_0) + c_0^m + c_\phi^m f_0 = 1.$$

The aggregate demand of investors is given by

$$X(p, \phi) = \pi_1 a(p) + \pi_3 b(p) + \pi_2 X^u + \sigma_\xi \left(\Phi'\left(\frac{a(p) - X^u}{\sigma_\xi}\right) - \Phi'\left(\frac{b(p) - X^u}{\sigma_\xi}\right)\right).$$

Finally, the informational efficiency in the constrained economy is same as in unconstrained

economy i.e., $\beta^c = \beta^u$. All the coefficients are reported in the Appendix.

Proof. We derive two results here: (1) the price is informationally equivalent to $\phi = v - \frac{z}{\beta^c}$ with $\beta^c = \beta^u$, and (2) the expression of $f'(p)$. The rest of the results are derived in the Appendix.

By the exact law of large numbers one can write the aggregate demand of investors as

$$X(p, \phi) = \int x_i di = E_{\xi_i} \left[T \left(X^u(p, \phi) + \xi_i; a(p), b(p) \right) \right]$$

For a given price p, the aggregate demand in constrained economy X is an increasing (and thus invertible) function of aggregate demand in unconstrained economy X^u. Therefore given p, one can easily compute $X = 1 - x_m(p)$, from which one can infer X^u, from which, in turn, one can express $\phi^u = v - \frac{z}{\beta^u}$. This proves the first result.

It remains to find the function $f'(p)$. Differentiating the market-clearing condition (i.e., $X(p, \phi) + x_m(p) = 1$) implicitly, we have

$$f'(p) = \frac{d\phi}{dp} = -\frac{\frac{\partial}{\partial p}(X(p, \phi) + x_m(p))}{\frac{\partial}{\partial \phi}(X(p, \phi) + x_m(p))}.$$

For the numerator, we have

$$\frac{\partial}{\partial p}(X(p, \phi) + x_m(p)) = c_p^m + \pi_2 c_p - \pi_1 a'(p) - \pi_3 b'(p).$$

The sensitivity of aggregate demand with respect to p comes from four sources. First, there is a market maker, who contributes c_p^m. Second, there is a fraction π_2 of unconstrained investors, each contributing c_p. Third, there is a fraction π_1 of investors whose lower constraint $a(p)$ binds and each of them adjusts their demand by $a'(p)$. Finally, the fraction π_3 of investors whose upper constraint binds adjusts their demand by $b'(p)$.

By a similar argument, for the denominator, we have

$$\frac{\partial}{\partial \phi}(X(p, \phi) + x_m(p)) = c_\phi^m + \pi_2 c_\phi.$$

Finally, we are left to determine the fractions π_1, π_2 and π_3. The fraction of investors constrained by lower constraints, π_1, is given by

$$\pi_1(p, \phi) = \mathrm{P}\left(x_i < a(p)\right) = \mathrm{P}\left(X^u(p, \phi) + \xi_i < a(p)\right) = \Phi\left(\frac{a(p) - X^u(p, \phi)}{\sigma_\xi}\right),$$

where $\Phi(\cdot)$ denotes the CDF of a standard normal random variable. The expressions for π_2 and π_3 can be derived analogously. ∎

Proposition 2 is our first main result which establishes the existence of a tractable, generalized linear equilibrium in an REE model with portfolio constraints, even when price may not be linear. Moreover, it states that for an exogenously given signal precision τ_ϵ, portfolio constraints are irrelevant for informational efficiency of prices.

Observation 1: *With exogenous information, constraints do not affect price informativeness.*

The economics of our irrelevance result sheds lights on how price aggregates information in an economy with portfolio constraints. In essence, the aggregate investors' demand (and hence the market-clearing price) varies with and reflects fundamentals via the change in *fractions* of constrained investors. Consider an improvement in the asset fundamental v (while fixing the endowment shock z). Each investor would like to demand more of the risky asset. While some investors could not increase their demand due to the upper portfolio constraint, in aggregate more (fewer) investors become constrained by the maximal long (short) positions. Thanks to the Exact Law of Large Numbers, the aggregate demand increases almost surely, hence revealing the improvement of asset fundamentals via a higher market-clearing price.[6]

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[6]Our irrelevance result relates to but differs from the irrelevance result in a recent paper by Dávila and Parlatore (2017). Instead of portfolio constraints, Dávila and Parlatore (2017) studies the impact of various forms (quadratic, linear, or fixed) of trading cost on informational efficiency. They find that when investors are

Observation 2: *There could be multiple equilibria depending on the sign of $f'(p)$.*

If $f(p)$ is monotonic, there is a unique financial market equilibrium i.e., every realization of fundamentals (v, z) can be supported by only one price. If $f(p)$ is non-monotonic, then some realization of fundamentals can be supported by two prices which implies that there could be multiple equilibria. Note from equation 6 that $f'(p)$ has some positive and negative terms, which could make $f'(p)$ change its sign and hence non-monotonic. Whether we have multiple or unique equilibria is driven by the type of constraints. In Section 3, we focus on margin requirements and show that in that case, the equilibrium is always unique. In Appendix B, we study a form of borrowing constraint as in Yuan (2005) and characterize the conditions for multiple equilibria.

2.3 Value of information

After solving for the financial market equilibrium at $t = 1$, in this section, we study how the incentives of investors to acquire information at $t = 0$ are affected by portfolio constraints. We will first derive the expression for marginal value of information under general portfolio constraints and then show that an investor's marginal value of information decreases if his constraints are tightened.

At date 0, we assume that investors preferences are given by

$$U_0 = E_0 \left[u_0 \left(E_1 \left[u_1 \left(W_2 \right) \right] \right) \right]$$

ex ante homogeneous, trading cost reduces *each* investor's trading incentives for both information and hedging motives symmetrically. Thus, in equilibrium, the signal-to-noise ratio of price is unaffected. Our irrelevance result relates to theirs as our portfolio constraints can be formulated as a specific form of price-dependent trading cost: zero when the trading amount is in $[a(p), b(p)]$ and prohibitively high otherwise (assuming investors' endowment of the risky asset is zero). Importantly, however, portfolio constraints differ from trading costs because ex post it affects some investors' trading but not the others. The logic of our irrelevance result is therefore different from theirs and relies on the Exact Law of Large Numbers.

The inner utility function u_1 governs the standard risk aversion over terminal wealth. If the outer utility function u_0 is linear, then the investor is expected utility maximizer and is indifferent about timing of resolution of unceatinty. If the outer utility function is concave, then the investor has preferences for early resolution of uncertainty. Refer to Van Nieuwerburgh and Veldkamp (2010) for more discussion on this.

At date 0, each investor chooses the precision of his private signal τ_{ϵ_i} to maximize his expected utility:

$$E_0\left[u_0\left(-E_1\left\{\exp(-\gamma(W_0 + x_i(v + \theta - p) + e_i\theta + \eta_i - C(\tau_{\epsilon_i})))\right\}\right)\right] \quad \text{where} \quad x_i = T(x_i^u; a(p), b(p))$$

where $C(\cdot)$ is the cost of acquiring information. The certainty equivalent at time 1 can be written as

$$CE_1 = W_0 + x_i(E[v|\mathcal{F}_i] - p) - \frac{\gamma}{2\tau_{v,i}}x_i^2 - \frac{\gamma}{2\tau_\theta}\left(x_i + e_i\right)^2 - \frac{\gamma}{2\tau_\eta} - C(\tau_{\epsilon_i}). \tag{7}$$

where $\tau_{v,i}^{-1} = var(v|F_i)$. Next, note that the demand in the unconstrained setting is

$$x_i^u = \frac{\tau_i}{\gamma}(E[v|\mathcal{F}_i] - p - \gamma e_i\tau_\theta^{-1}) \Rightarrow E[v|\mathcal{F}_i] - p = \frac{\gamma}{\tau_i}x_i^u + \frac{\gamma}{\tau_\theta}e_i.$$

Substituting this into the certainty equivalent in equation 7, we get

$$CE_1 = \underbrace{-\frac{\gamma}{2\tau_i}\left(x_i^u - x_i\right)^2}_{\text{the term due to constraints}} + W_0 + \frac{\gamma}{2\tau_i}\left(x_i^u\right)^2 - \frac{\gamma}{2\tau_\theta}e_i^2 - \frac{\gamma}{2\tau_\eta} - C(\tau_{\epsilon_i})$$

One can see that the change in the time-1 certainty equivalent due to introduction of constraints is captured by a single term, $-\frac{\gamma}{2\tau_i}\left(x_i^u - x_i\right)^2$, which captures the "distance" between the demand with and without constraints. It is immediate to see that CE_1 decreases as constraints become tighter, everything else being equal.

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2.3.1 Marginal value of information

We calculate the marginal value of information (in closed form) under two specifications of date-0 utility function: $u_0(x) = x$ and $u_0(x) = -\frac{1}{\gamma}\log(-x)$. All our results remain qualitatively the same under both specifications.

Case 1. $u_0(x) = x$: In this case, investors have expected utility preferences and are indifferent about the timing of resolution of uncertainty. Define date-0 certainty equivalent as the solution to $e^{-\gamma CE_0} = \mathbb{E}_0[e^{-\gamma CE_1}]$ or, equivalently, $CE_0 = \frac{-\ln \mathbb{E}_0[e^{-\gamma CE_1}]}{\gamma}$.

Case 2. $u_0(x) = -\frac{1}{\gamma}\log(-x)$: In this case, investors have preference for early resolution of uncertainty. The date-0 certainty equivalent is given by $CE_0 = E[CE_1]$.

We define the *marginal value of information* as $\frac{dCE_0}{d\tau_{\epsilon_i}}$. In the next proposition, we characterize the marginal value of information under general portfolio constraints in both cases and show that the marginal value of information decreases when an investor's constraints tighten.

Proposition 3. *(Marginal value of information) The marginal value of information for an investor i*

- *in case 1 is given by*

$$\text{MVI} = \frac{\tau_i}{2\tau_{v,i}^2\gamma} \qquad \underbrace{\frac{U_0^u}{U_0}}_{\text{the term due to constraints}} \tag{8}$$

- *in case 2 is given by*

$$\text{MVI} = \frac{\gamma}{2\tau_{v,i}^2}\mathbb{E}[x_i^2] + \frac{\tau_i}{2\gamma\tau_{v,i}^2}\pi_{2,i} \tag{9}$$

where $\tau_{v,i} = \tau_{\epsilon_i} + \tau_v + \beta^2(\tau_u + \tau_z)$ is the total precision of information about learnable component to investor i; $\tau_i = \frac{\tau_{v,i}\tau_\theta}{\tau_{v,i}+\tau_\theta}$ is the total precision of fundamentals; $U_0^u = E[-e^{-\gamma CE_1}\mathbb{I}(x_i^u = x_i)]$ is the expectation of utility in the states when constraints do not bind; $U_0 = E[-e^{-\gamma CE_1}]$ is time-0

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expected utility; x_i is investor i's demand of risky asset in the constrained economy and $\pi_{2,i}$ denotes investor i' probability of being unconstrained, for a given τ_{ϵ_i}.

In both cases, the marginal value of information decreases when individual investor's constraints become tighter, holding everything else fixed.

Proposition 3 shows clearly how portfolio constraints affect an investor's incentives to acquire information. In case 1, the effect is captured by the term $\frac{U_0^u}{U_0} \in [0, 1]$: when constraints almost always (never) bind, this term is close to zero (one). In case 2, when constraints tighten, in the sense that $a(p)$ and $b(p)$ becomes closer to zero, both $\mathbb{E}[x_i^2]$ and $\pi_{2,i}$ in equation 9 decrease. In other words, the tightening of an investor's portfolio constraints in general weakens his incentive to acquire information. Intuitively, information is more valuable to an investor when he can profit more from it.[7]

Next, we study how the equilibrium information acquisition changes when all investors' portfolio constraints become tighter. Tightening constraints for all investors, nonetheless, is more complicated because the equilibrium price distribution will change due to the market maker's risk aversion, in turn affecting the price-dependent constraints. In the case with riskneutral market maker, we can prove that tightening constraints for all investors reduces each investor's marginal value of information. We state our results formally in the following proposition.

Proposition 4. *Suppose that all investors face portfolio constraints $a(p)$ and $b(p)$ and the market maker is risk-neutral. If constraints become tighter for all investors, i.e. $a(p)$ and $b(p)$ becomes $\hat{a}(p)$ and $\hat{b}(p)$ such that $\forall p$, $\hat{b}(p) \leq b(p)$ and $\hat{a}(p) \geq a(p)$, equilibrium information acquired is lower and prices become less informative.*

Proposition 4 illustrates one of the key force of our mechanism: tighter constraints could reduce investors' incentive to acquire information. Therefore, even though constraints do not

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[7] Note that in case 2, even if the investor is always constrained, i.e. $\pi_{2,i} = 0$, the marginal value of information is still positive (as long as $\mathbb{E}[x_i^2] > 0$) because the investor has preferences for early resolution of uncertainty.

affect the information aggregation channel, they affect information acquisition incentives and hence informational efficiency of prices.

To close the model with the characterization of equilibrium investors' information acquisition, we shall need to further specify the nature of the portfolio constraints. In the rest of the paper, we shall focus on margin requirements and study their interactions with informational efficiency of price.

3   Portfolio constraints from margin requirements

So far we have studied general, price-dependent portfolio constraints. In this section, we apply our model to study constraints that arise from margin requirements. In order to demonstrate the main mechanism of the paper analytically, namely, the interaction between funding constraints and informational efficiency, throughout this section we assume the market maker to be risk neutral, i.e. $\gamma_m = 0$. In Section 4, we shall relax this assumption and numerically illustrate our model's implications on asset prices. Also for simplicity, we assume the investors' time-0 preference as $u_0(x) = -\frac{1}{\gamma}\log(-x)$, i.e. the second case in Proposition 3. Our results also hold in the case with expected utility preference $u_0(x) = x$.

3.1   Funding constraints from margin requirements

Our notion of margin requirements is standard and follows closely from Brunnermeier and Pedersen (2009). Specifically, to build a long position of the risky asset, an investor can borrow from a financier at the risk-free rate but he has to pledge cash margin $m^+(p) \geq 0$ per unit of asset to the financier as collateral. Similarly, to establish a short position an investor has to provide cash margin $m^-(p)$ per unit of asset as collateral. Therefore, investors face a funding

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constraint that the total margin on their positions cannot exceed their initial wealth :

$$m^-(p)[x_i]^- + m^+(p)[x_i]^+ \leq W_0,$$

where $[x_i]^-$ and $[x_i]^+$ are the positive and negative parts of x_i respectively. We can re-write the margin requirements in the form of portfolio constraints as

$$a(p) = -\frac{W_0}{m^-(p)}, \quad b(p) = \frac{W_0}{m^+(p)} \tag{10}$$

Equation 10 shows that an investor faces tighter constraints when his initial wealth is lower and/or the margin requirements set by the financier are higher. We shall delay the discussion of how margins are set by the financier to Section 3.2. For now we assume that margins do not depend on the price and will later prove that this is indeed the case in equilibrium with risk-neutral market maker.

We proceed with solving the model under margin requirements backwards. The financial market equilibrium at $t = 1$ is just a special case of Proposition 2 and consequently, the next result is a straightforward extension.

Corollary 1. *Suppose investors face margin constraints given by 10 and have identical signal variances τ_ϵ^{-1}. Then there exists an equilibrium with informational efficiency β^c and a unique function $f(p) = \phi \equiv v - (\beta^c)^{-1}z$ which satisfies the ODE*

$$f'(p) = \frac{c_p^m + \pi_2 c_p}{c_\phi^m + \pi_2 c_\phi}. \tag{11}$$

In addition, the equilibrium is always unique.

Next, we characterize the equilibrium at $t = 0$, i.e. the information acquisition stage. In any symmetric equilibrium, investors acquire information until the marginal cost of doing so equals the marginal value of information. The result below follows directly from Proposition 3

and [4].

Corollary 2. *Equilibrium information acquisition (τ_ϵ^*) at $t = 0$ satisfies*

$$C'(\tau_\epsilon^*) = \frac{\gamma}{2\tau_v^2}\mathbb{E}[x_i^2] + \frac{\tau}{2\gamma\tau_v^2}\pi_2. \tag{12}$$

In addition, τ_ϵ^* (and hence informational efficiency β^c) decreases when

- *initial wealth W_0 drops*

- *margins m^+ and m^- increase*

Corollary 2 implies that wealth effect arises endogenously in our model with constraints. As the investors' initial wealth decrease, they become more constrained thus acquire less information, reducing price informativeness in equilibrium. It is well-known that there is no wealth effect in a typical CARA-Normal REE model. To the best of our knowledge, ours is the first noisy REE model that admits closed-form solutions with margin constraints and have wealth effects.

3.2   Endogenous margin requirements

Up until now we have assumed that margins are exogenous. In this section, we endogenize margins as in Brunnermeier and Pedersen (2009). We assume the financiers set margin in order to control their Value-at-Risk (henceforth VaR):

$$m^+(p) = \inf\{m^+(p) \geq 0 : Pr(p - v > m^+(p)|p) \leq 1 - \alpha\}$$
$$m^-(p) = \inf\{m^-(p) \geq 0 : Pr(v - p > m^-(p)|p) \leq 1 - \alpha\} \tag{13}$$

$m^+(p)$ and $m^-(p)$ are the margins on long and short positions (per unit of asset) respectively. Intuitively, the financiers require the investors to set aside a minimum amount of cash,

i.e. margin, which is just large enough to sufficiently cover the potential loss from trading with probability α. We assume that financier is uninformed but can set margin condition on prices. As detailed in the Appendix A of Brunnermeier and Pedersen (2009), this margin specification is motivated by the real-world margin constraints faced by hedge funds and capital requirements imposed on commercial banks. Note that we allow margins to depend on price and will later show that, in equilibrium they do not depend on prices.

Formally, our financial market equilibrium with endogenous margin constraints is defined as follows: (1) financiers determine the margin requirements with a conjectured price function; (2) investors and the market maker optimally choose their demand given the conjectured price function; (3) in equilibrium, the conjectured price function is consistent with market clearing. As before, we take the precisions of investors' signals as given.

Proposition 5. *(Financial market equilibrium under endogenous margin requirements) When the portfolio constraints are of the form of margin as in equation (10) and margins are endogenously determined by Value-at-Risk as in equation (13), there exists a unique generalized linear equilibrium. Moreover, the function $f(p)$, i.e. the sufficient statistic ϕ, is increasing in price.*

In the previous section, we show that exogenous funding constraints (from margins) affect informational efficiency. Now, we show how informational efficiency affects margin constraints.

We proceed with deriving the expression for margins. Since the market maker is risk-neutral, $p = E[v|p]$. For $m^+(p)$, we first determine the functions $m_n^+(p)$ that satisfies

$$
\begin{aligned}
1 - \alpha &= Pr(E[v|p] - v > m_n^+(p)|p) \\
&= Pr\left(\sqrt{\tau_m}(E[v|p] - v) > \sqrt{\tau_m}(m_n^+(p))\big|p\right) \\
&= 1 - \Phi\left(\sqrt{\tau_m}(m_n^+(p))\right)
\end{aligned}
$$

where $\tau_m = \mathbb{V}(f - p|p)$. Thus, we find

$$m^+(p) = [m_n^+(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}}\right]^+ \tag{14}$$

Similarly, one can define $m_n^-(p)$ which satisfies $Pr(v - p > m_n^-(p)|p) = 1 - \alpha$ and get

$$m^-(p) = [m_n^-(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}}\right]^+ \tag{15}$$

Note that margins are in fact independent of the level of prices and determined by two variables. Both margins on long and short positions increase in the exogenous level of confidence α and decrease in the endogenous informational efficiency of price β (through $\tau_m^{-1} = (\tau_v + \beta^2 \tau_z)^{-1} + \tau_\theta^{-1}$). We would like to emphasize the latter result that informational efficiency of price affects the tightness of margin constraints.

Proposition 6. *(Informational efficiency affects constraints) Suppose the market maker is risk-neutral. For a given investors' wealth W_0, when informational efficiency (β) decreases, margins (both m^+ and m^-) increase. This implies that the lower constraint (a) increases and the upper constraint (b) decreases. In other words, as informational efficiency drops, constraints become tighter.*

The proposition above establishes the last two links in the Figure 1. The intuition is as follows. Financiers use information in price to help to asses the risk that the loss from financing exceeds the margin. With less informative prices, financiers face more uncertainty about fundamentals and thus perceive higher risk of financing a trade and require higher margins.

Observation: *Informational efficiency affects constraints even if financiers do not learn from prices.* We would like to emphasize that our results do not rely on financiers' learning from prices. Indeed, one can compute the *unconditional* variance of returns as

$$\mathbb{V}[f - p] = \mathbb{E}[\mathbb{V}[f - p|p]] + \mathbb{V}[\mathbb{E}[f - p|p]] = \mathbb{E}[\mathbb{V}[f - p|p]]$$

For a given equilibrium signal precision τ_ϵ^*, the conditional variance $\mathbb{V}[f - p|p]$ is a constant thus equals to the unconditional variance $\mathbb{V}[f - p]$. This implies that even if the financier does not condition on the prices, he will set the same margins.

Amplification

In Section 3.1, we did a partial equilibrium analysis and argued that given margins, tighter funding constraints (a decrease in wealth) leads to lower informational inefficiency. In section 3.2, we argued that, given a level of wealth, lower informational inefficiency leads to tighter margins. Putting all the links together, we will have an amplification loop represented in Figure 1. We call it the *information spiral*. The main implication of the information spiral is that small changes in underlying funding constraints can lead to sharp reductions in information production and hence, informational efficiency. In the next section, we discuss the implications of this feedback loop for asset prices.

4   Asset pricing implications

In this section, we will derive the implications of a drop in wealth on the equilibrium risk premium and volatility of the risky asset. To do this, we relax the assumption that the market maker is risk-neutral. In Appendix C, we describe how financiers determine margins and characterize the financial market equilibrium with a risk-averse market maker. The main result of this section is that a drop in wealth could lead to large increases in risk premium, volatility and Sharpe ratio.

4.1 Risk premium

By definition, conditional risk premium is given by

$$rp(p) = \mathbb{E}[v - p|p] = \frac{\gamma_m}{\tau_m} x_m = \frac{\gamma_m}{\tau_m}(c_0^m + c_\phi^m f(p) - c_p^m p).$$

Hence the unconditional risk premium is given by

$$\mathbb{E}[v - p] = \mathbb{E}[rp(p)] = \frac{\gamma_m}{\tau_m}(c_0^m + c_\phi^m \bar{v} - c_p^m \mathbb{E}(p)).$$

Note that price is a non-linear function of fundamentals and hence we proceed with numerical analysis.

Figure 2: Risk premium

The figure plots risk premium as a function of precision of investors signal for different levels of wealth: $w = 0.2$ and $w = 2$. Other parameter values are set to: $\tau_u = \tau_z = \tau_v = \tau_\epsilon = 1$, $\bar{v} = 4$, $\gamma = 3$, $\gamma_m = 1.5$.
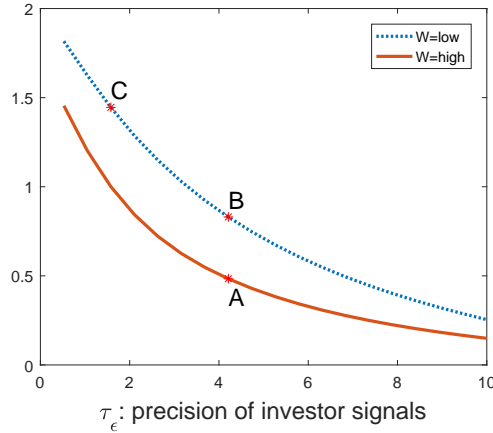

Figure 2 plots the unconditional risk premium in our model against τ_ϵ, precision of investors' signals for two different levels of wealth. Suppose that the point A corresponds to the equilibrium with high wealth level. At this wealth, most of the investors are unconstrained and the equilibrium is close to the unconstrained setting. Now consider a negative shock to

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investors' wealth. With a decreased wealth, investors' capacity to go long or short the asset is diminished due to tighter constraints, which has a similar effect to lowering their risk-bearing capacity (increasing risk aversion). Therefore, the risk premium rises. This argument implies that absent the information production channel (i.e., holding τ_ϵ fixed), the wealth drop would cause an increase in risk premium that corresponds to the move from solid line (corresponding to high wealth level) to dashed line (corresponding to low wealth level), from point A to point B. Moreover, because of the information spiral, investors in equilibrium acquire less information, which leads to an additional increase in risk premium, corresponding to the move from point B to point C along the dashed line. Thus, an effect of drop in wealth on risk premium is amplified through the information production channel and the equilibrium moves from point A to point C.

4.2 Volatility of returns

Next, we turn to volatility of returns. Note that variance of returns can be written as

$$\mathbb{V}[f - p] = \mathbb{V}[\mathbb{E}(f - p|p)] + \mathbb{E}[\mathbb{V}[f - p|p]] \tag{16}$$

$$= \mathbb{V}[rp(p)] + \mathbb{E}[\mathbb{V}[v|p]] + \tau_\theta^{-1} \tag{17}$$

$$= \left(\frac{\gamma_m}{\tau_m}\right)^2 \mathbb{V}[x_m(p)] + (\tau_v + \beta^2 \tau_z)^{-1} + \tau_\theta^{-1} \tag{18}$$

Because of the non-linearity of the conditional risk premium $(rp(p))$, the above expression cannot be simplified further. Hence, we proceed numerically.

Panel (a) of Figure 3 plots volatility of returns against τ_ϵ, precision of investors' signals for two different levels of wealth. Suppose point A is an equilibrium with high wealth level. Now imagine lowering wealth. We see that, holding τ_ϵ fixed, as wealth drops, volatility decreases (corresponding to the move from point A to point B). The intuition can be seen from equation 18. As wealth decreases, the second term does not change with fixed τ_ϵ. The first term

Figure 3: Price volatility

The figure plots price volatility as a function of precision of investors signal for different levels of wealth, $w = 0.2$ and $w = 2$ (panel (a)) and different levels of parameter α, $\alpha = 0.95$ and $\alpha = 0.99$ (panel (b)). Other parameter values are set to: $\tau_u = \tau_z = \tau_v = \tau_\epsilon = 1$, $\bar{v} = 4$, $\gamma = 3$, $\gamma_m = 1.5$.

(a)

(b)
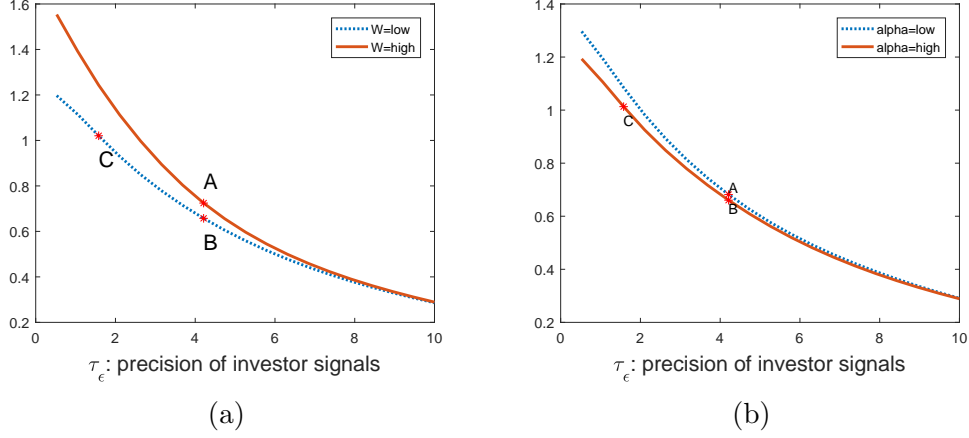
decreases because of lower volatility of investors' (and hence market maker's) position. This is the direct effect. However, since investors acquire less information when they are constrained, we have an increase in volatility corresponding to the move from point B to point C. Therefore, the indirect direct operating through the information spiral may dominate so that volatility increases as wealth drops.

We also examine the effects of margin requirements (as measured by parameter α) on price volatility. It has long been argued that tighter margin requirements could stabilize prices. The argument is that tighter margin requirements should curb the investors' positions therefore limiting the price impact of their information and liquidity shocks. Consider panel (b) of Figure 3. As margin constraints tighten, volatility indeed drops as we move from point A to point B, confirming the conventional wisdom. This result holds with τ_ϵ fixed (direct effect). However, since investors acquire less information with tighter constraints, the volatility may increase with tighter funding requirements (we move from point B to point C). Thus, our model provides an alternative explanation for why tightening of margin requirements can increase volatility, complementing the results of Wang (2015).

4.3 Sharpe ratio

Finally, we examine the Sharpe ratio of the risky asset. The Sharpe ratio is defined as $SR = \frac{\mathbb{E}[v-p]}{\mathbb{V}[v-p]}$. With τ_ϵ fixed, we argued in the previous subsections that risk premium rises and volatility drops (point A to point B in both plots) as the wealth of investors decreases. This implies that Sharpe ratio rises as wealth drops, holding τ_ϵ fixed. Now, with endogenous margins and information acquisition, we argued in previous section that risk premium and volatility both rise. This implies that the indirect effect cannot be signed. In Figure 4, we see that both direct effect(A to B) and indirect effect (B to C) are in the same direction, amplifying the effect of the wealth shock.

Figure 4: Sharpe ratio

The figure plots Sharpe ratio as a function of precision of investors signal for different levels of wealth, $w = 0.2$ and $w = 2$. Other parameter values are set to: $\tau_u = \tau_z = \tau_v = \tau_\epsilon = 1$, $\bar{v} = 4, \gamma = 3, \gamma_m = 1.5$.
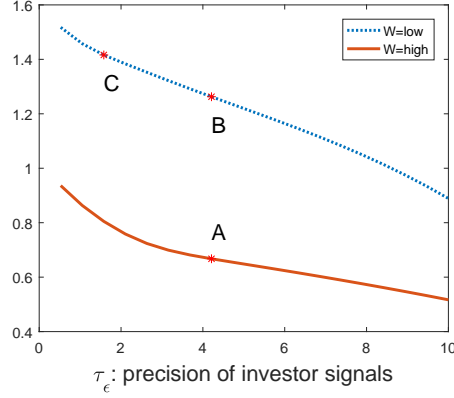
5 Conclusion

In this paper we developed a tractable REE model with general portfolio constraints. We applied our methodology to study a model with endogenous margins constraints. We uncovered a novel amplification mechanism, which we call the information spiral. A drop in investors' wealth tightens constraints and reduces their incentive to acquire information, which lowers price informativeness. Moreover, financiers who use information in prices to assess the risk of financing a trade face more uncertainty and set tighter margins, which further tightens constraints. This implies that risk premium, conditional volatility and Sharpe ratios rise disproportionately as investors' wealth drops. These results imply a new, information-based rationale why the wealth of investors is important.

Our information spiral can also potentially generate complementaries in information acquisition: as other investors acquire more information, margins become less tight, giving incentives to a particular investor to acquire more information. This provides a mechanism for complementaries in information acquisition alternative to that in Goldstein and Yang (2017). We explore implications of the above complementaries in the ongoing work.

While we have assumed normal distribution for the random variables in the model, our results can be generalized to distributions within the exponential family, as in Breon-Drish (2015). We leave it for the future work.

6 Appendix

Proof. (Proposition 1) The first order condition for investor i is given by

$$x_i = \frac{\tau}{\gamma} \left(E\left(v|F_i\right) - p - \gamma e_i \tau_\theta^{-1} \right) \quad \text{where} \quad \tau^{-1} = Var\left[v + \theta|F_i\right]$$

and the first order condition for the market maker is $x_m = \tau_m \frac{E(v|p)-p}{\gamma_m}$. Using the standard normal updating, we can write

$$E\left(\tilde{v}|F_i\right) = \frac{\tau_\epsilon \tilde{s}_i + \beta^2 \left(\tau_u + \tau_z\right)\tilde{\phi} + \beta\tau_u e_i}{\tau_\epsilon + \beta^2 \left(\tau_u + \tau_z\right) + \tau_v} \quad \text{and} \quad \frac{1}{\tau} = \frac{1}{\tau_\epsilon + \beta^2 \left(\tau_u + \tau_z\right) + \tau_v} + \frac{1}{\tau_\theta}$$

where varibles with $\tilde{}$ denote demeaned variables. Substituiting these into the market clearing condition, we get

$$\int \frac{\tau}{\gamma} \left(E\left(v|F_i\right) - p - \gamma e_i \tau_\theta^{-1} \right) di + \frac{\tau_m}{\gamma_m} \left(E\left(v|p\right) - p \right) = 1.$$

This implies that

$$\frac{\tau}{\gamma} \left(\frac{\tau_\epsilon v + \beta^2 \left(\tau_u + \tau_z\right)\phi + \beta\tau_u z}{\tau_\epsilon + \beta^2 \left(\tau_u + \tau_z\right) + \tau_v} \right) - \frac{\tau}{\tau_\theta} z + \frac{\tau_m}{\gamma_m} \frac{\beta^2 \tau_z \phi}{\beta^2 \tau_z + \tau_v} = p \left(\frac{\tau}{\gamma} + \frac{\tau_m}{\gamma_m} \right).$$

This implies that β satisfies

$$-\frac{1}{\beta} = \frac{\left(\beta\tau_\theta\tau_u - \gamma\left(\tau_\epsilon + \beta^2 \left(\tau_u + \tau_z\right) + \tau_v\right)\right)}{\tau_\theta\tau_\epsilon}$$

$$\beta^3\gamma\left(\tau_u + \tau_z\right) - \beta^2\tau_u\tau_\theta + \beta\gamma\left(\tau_\epsilon + \tau_v\right) - \tau_\theta\tau_\epsilon = 0 \tag{19}$$

First note that the solution of this equation is always positive and there exists atleast one solution. The solution is unique if the slope of the above polynomial is either always positive

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or always negative. The slope of above equation is

$$3\beta^2\gamma\left(\tau_u+\tau_z\right)-2\beta\tau_u\tau_\theta+\gamma\left(\tau_\epsilon+\tau_v\right)$$

At $\beta=0$, the slope is positive and the slope is always positive if the above equation has no roots. This is true iff

$$\tau_u^2\tau_\theta^2<3\gamma^2\left(\tau_u+\tau_z\right)\left(\tau_\epsilon+\tau_v\right)$$

Finally, using implicit differentiation of 19, β increases in τ_ϵ iff $\tau_\theta-\beta\gamma>0$. Equation 19 can be rewritten as

$$\beta^3\gamma\tau_z+\beta^2\tau_u\left(\beta\gamma-\tau_\theta\right)+\beta\gamma\tau_v+\left(\beta\gamma-\tau_\theta\right)\tau_\epsilon=0$$

From above equation, it is obvious that $\tau_\theta-\beta\gamma>0$ and hence β increases in τ_ϵ. Since the aggregate demand of investors and market makers can depend on v only through ϕ we find

$$c_\phi=\frac{\tau}{\gamma}\frac{\partial E[v|\mathcal{F}_i]}{\partial v}=\frac{\tau}{\gamma}\left(\frac{\tau_\epsilon+\beta^2\left(\tau_u+\tau_z\right)}{\tau_\epsilon+\beta^2\left(\tau_u+\tau_z\right)+\tau_v}\right),$$

$$c_\phi^m=\frac{\tau_m}{\gamma_m}\frac{\partial E[v|\mathcal{F}_m]}{\partial v}=\frac{\tau_m}{\gamma_m}\frac{\beta^2\tau_z}{\beta^2\tau_z+\tau_v}.$$

Similarly,

$$c_p=\frac{\tau}{\gamma},\quad c_p^m=\frac{\tau_m}{\gamma_m}.$$

Finally,

$$\xi_i=\frac{\tau}{\gamma}\left(\frac{\tau_\epsilon\tilde{\epsilon}_i+\beta\tau_u u_i}{\tau_\epsilon+\beta^2\left(\tau_u+\tau_z\right)+\tau_v}-\gamma u_i\tau_\theta^{-1}\right)$$

$$\sigma_\xi^2=\left(\frac{\tau}{\gamma}\right)^2\left(\frac{\tau_\epsilon\tilde{\epsilon}_i+\beta\tau_u u_i}{\tau_\epsilon+\beta^2\left(\tau_u+\tau_z\right)+\tau_v}-\gamma u_i\tau_\theta^{-1}\right)$$

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■ Proof. (Proposition 2) We prove that for every p there exists unique $\phi = f(p)$ such that market clears. Indeed, the market clearing can be written as

$$X(\phi, p) + c_0^m - c_p^m p + c_\phi^m \phi = 1.$$

For a given p, aggregate investors' demand $X(\phi, p)$ is monotone in ϕ. Thus, there is at most one solution. At least one solution exists by the Intermediate Value Theorem. The aggregate demand at $+\infty(-\infty)$ is equal to $+\infty(-\infty)$, thus at some intermediate point aggregate demand has to be equal to 1.

We compute a closed-form expression for the aggregate demand of investors $X(\phi, p)$. It can be split into three parts. For a fraction π_1 of investors the lower constraint $a(p)$ will bind. The latter fraction can be calculated as follows

$$\pi_1(\phi, p) = Pr\left(X^u(p, \phi) + \xi_i < a(p)\right) = \Phi\left(\frac{a(p) - X^u(p, \phi)}{\sigma_\xi}\right),$$

where $\Phi(\cdot)$ denotes the CDF of a standard normal random variable. They will contribute $\pi_1(\phi, p)a(p)$ to the aggregate demand. Similarly, a fraction π_3 of investors for whom the upper constraint $b(p)$ binds will contribute $\pi_3(\phi, p)b(p)$, where

$$\pi_3(\phi, p) = 1 - \Phi\left(\frac{b(p) - X^u(p, \phi)}{\sigma_\xi}\right).$$

Finally a fraction $\pi_2(\phi, p) = 1 - \pi_1 - \pi_3$ will be unconstrained. They will contribute $\pi_2 \cdot (X^u + E[\xi_i | (\xi_i + X^u) \in [a(p), b(p)]])$. The last term can be further simplified to

$$\pi_2 E[\xi_i | (\xi_i + X^u) \in [a(p), b(p)]] = \sigma_\xi\left(\Phi'\left(\frac{a(p) - X^u}{\sigma_\xi}\right) - \Phi'\left(\frac{b(p) - X^u}{\sigma_\xi}\right)\right).$$

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Combining all of the terms we get

$$X(\phi, p) = \pi_1 a(p) + \pi_3 b(p) + \pi_2 X^u + \sigma_\xi \left(\Phi' \left(\frac{a(p) - X^u}{\sigma_\xi} \right) - \Phi' \left(\frac{b(p) - X^u}{\sigma_\xi} \right) \right).$$

■

Proof. (Proposition 3) The key to computing MVI is to compute $\frac{dU_0}{d\tau_{\epsilon i}} = \frac{d}{d\tau_{\epsilon i}} E[-e^{-\gamma CE_1}]$. We note that CE_1 depends on the realizations of random variables $(s_i = v + \epsilon_i, e_i, p)$ of which only the distribution of ϵ_i is affected by $\tau_{\epsilon i}$. To emphasize the latter fact we'll use the notation $\epsilon i(\tau_{\epsilon i})$. The key step in the proof is to substitute $\epsilon_i(t) = \frac{1}{t} B_t$, where B_t is a Brownian motion that is independent of all other variables in the model. Indeed, such a substitution is valid, as ex-ante both $\epsilon_i(t)$ and $\frac{1}{t} B_t$ have the same distribution $(N(0, 1/t))$. Hence, computing $E[-e^{-\gamma CE_1}]$, with or without substitution of $\epsilon_i(t) = \frac{1}{t} B_t$, will produce the same result. The dependence of CE_1 on $\tau_{\epsilon i}$ comes only through its' dependence on B_{τ_ϵ} and we emphasize this fact by writing $CE_1(B_{\tau_\epsilon})$. The advantage of substitution we've made is that now we can utilize Ito's lemma to compute $dCE_1(B_{\tau_{\epsilon i}})$. [8]

In particular, we proceed as follows:

$$\frac{d}{d\tau_{\epsilon i}} E[-e^{-\gamma CE_1}] = -E \left[\frac{de^{-\gamma CE_1(B_{\tau_{\epsilon i}})}}{d\tau_{\epsilon i}} \right].$$

_____

[8]There is a technicality here. Ito's lemma will be applicable to $CE_1(B_{\tau_{\epsilon i}})$ that is C^2 in $B_{\tau_{\epsilon i}}$. However, our function is only C^1 (continuously differentiable). One can also show that it is convex, which makes the Ito-Tanaka-Meyer rule applicable (see, e.g. Cohen, Elliot (2015), p. 352, also Bjork (2015) p. 18 for a more light reading). Since CE_1 is C^1 the local time terms in the Ito-Tanaka-Meyer rule disappear and we can write the Ito rule in the usual way. We now show that CE_1 is continuously differentiable.

$$CE_1(B_{\tau_{\epsilon i}}) = \begin{cases} -\frac{\gamma}{2\tau_i} (x_i^u - b)^2 + \frac{\gamma}{2\tau_i} (x_i^u)^2 + ... & \text{if } x_i^u > b(p) \\ \frac{\gamma}{2\tau_i} (x_i^u)^2 + ... & \text{if } x_i^u \in (a(p), b(p)) \\ -\frac{\gamma}{2\tau_i} (x_i^u - a)^2 + \frac{\gamma}{2\tau_i} (x_i^u)^2 + ... & \text{if } x_i^u < a(p) \end{cases}$$

$$CE_1'(B_{\tau_{\epsilon_i}}) = \begin{cases} \frac{\gamma}{\tau_i} b & \text{if } x_i^u > b(p) \\ \frac{\gamma}{\tau_i} x_i^u & \text{if } x_i^u \in (a(p), b(p)) \\ \frac{\gamma}{\tau_i} a & \text{if } x_i^u < a(p) \end{cases}$$

So we see that CE_1' is increasing and continuous implying that CE_1 is C^1 and convex.

We use Ito's lemma to compute

$$de^{-\gamma CE_1(B_{\tau_{\epsilon i}})} = -\gamma e^{-\gamma CE_1(B_{\tau_{\epsilon i}})} dCE_1 + \frac{\gamma^2}{2} e^{-\gamma CE_1(B_{\tau_{\epsilon i}})} dCE_1^2.$$

In order to compute $E[dCE_1]$ we use the law of iterated expectations and write

$$E[dCE_1] = E\left[E_{\tau_{\epsilon i}}[dCE_1]\right],$$

where $E_t[\cdot] = E[\cdot | v + \frac{1}{t} B_t, e_i, p]$. We first consider the case $v + \frac{1}{t} B_t, e_i, p$ are such that $x_i^u \in (a(p), b(p))$. In this case,

$$CE_1 = \frac{\tau_i}{2\gamma}\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)^2 + \ldots$$

$$dCE_1 = \frac{d\tau_i}{2\gamma}\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)^2 + \frac{\tau_i}{\gamma}\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right) dv_i + \frac{\tau_i}{2\gamma}\left(dv_i\right)^2.$$

$$(dCE_1)^2 = \left(\frac{\tau_i}{\gamma}\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)\right)^2 (dv_i)^2$$

where we've suppressed the terms unaffected by $\tau_{\epsilon i}$ and denoted $v_i = E[v|\mathcal{F}_i]$. We now compute

$$d\tau_i = d\left(\left(\frac{1}{\tau_{\epsilon i} + \beta^2\left(\tau_u + \tau_z\right) + \tau_v} + \frac{1}{\tau_\theta}\right)^{-1}\right)$$

$$= \frac{\tau_\theta^2}{\left(\beta^2(\tau_u + \tau_z) + \tau_{\epsilon i} + \tau_\theta + \tau_v\right)^2} d\tau_{\epsilon i}$$

$$= \left(\frac{\tau_i}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i},$$

where we denoted $\tau_{v,i} = \tau_{\epsilon i} + \beta^2\left(\tau_u + \tau_z\right) + \tau_v$.

Recall that

$$v_i = \mathbb{E}(v|\mathcal{F}_i)\frac{\tau_{\epsilon i} v + B_{\tau_{\epsilon i}} + \beta^2\left(\tau_u + \tau_z\right)\phi + \beta\tau_u e_i}{\tau_{v,i}}$$

Differentiating this expression, we get

$$dv_i = v \frac{d\tau_{\epsilon i}}{\tau_{v,i}} + \frac{dB}{\tau_{v,i}} + \frac{d\tau_{\epsilon_i}}{\tau_{v,i}} v_i$$

$$(dv_i)^2 = \left(\frac{dB}{\tau_{v,i}}\right)^2 = \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}.$$

Note that since $E_{\tau_{\epsilon i}}[v] = v_i$ and $E_{\tau_{\epsilon i}}[dB] = 0$, we have $E_{\tau_{\epsilon i}}[dv_i] = 0$. Hence,

$$E_{\tau_{\epsilon i}}[dCE_1] = \frac{d\tau_i}{2\gamma} E_{\tau_{\epsilon i}}\left[\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)^2\right] + \frac{\tau_i}{2\gamma} \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}.$$

Substituiting these expressions in 6, we get

$$E_{\tau_{\epsilon i}}\left[de^{-\gamma CE_1}\right] = -\gamma e^{-\gamma CE_1(B_{\tau_{\epsilon i}})} E_{\tau_{\epsilon i}}[dCE_1] + \frac{\gamma^2}{2} e^{-\gamma CE_1(B_{\tau_{\epsilon i}})} E_{\tau_{\epsilon i}}\left[dCE_1^2\right]$$

$$= e^{-\gamma CE_1}\left(\frac{-\left(\frac{\tau_i}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i} E_{\tau_{\epsilon i}}\left[\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)^2\right]}{2} - \frac{\tau_i}{2} \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2} + \frac{\gamma^2}{2}\left(\frac{\tau_i}{\gamma}\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right)\right)^2 \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}\right)$$

$$= -e^{-\gamma CE_1} \frac{\tau_i}{2} \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}$$

We now consider the case $v + \frac{1}{t} B_t, e_i, p$ are such that $x_i^u < a(p)$. In this case,

$$CE_1 = \frac{\gamma}{2\tau_i}(x_i^u)^2 - \frac{\gamma}{2\tau_i}(x_i^u - a(p))^2 + \text{some unrelated terms}$$

$$= \frac{\gamma}{2\tau_i}(2x_i^u - a(p))a(p)$$

$$= a(p)\left(v_i - p - \gamma e_i \tau_\theta^{-1}\right) - \frac{\gamma}{2\tau_i} a(p)^2.$$

Differentiating this expression, we get

$$dCE_1 = a(p)dv_i + \frac{\gamma}{2\tau_i^2} a(p)^2 d\tau_i.$$

$$= a(p)dv_i + \frac{\gamma}{2\tau_i^2} a(p)^2 \left(\frac{\tau_i}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i}$$

$$= a(p)dv_i + \frac{\gamma}{2} a(p)^2 \left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i}$$

For conditional expectation, we simply write

$$E_{\tau_{\epsilon i}}\left[dCE_1\right] = \frac{\gamma}{2}a(p)^2\left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i}.$$

$$(dCE_1)^2 = a(p)^2 \left(dv_i\right)^2$$
$$= a(p)^2 \frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}$$

Finally,

$$E_{\tau_{\epsilon i}}\left[de^{-\gamma CE_1(B_{\tau_\epsilon})}\right] = -\gamma e^{-\gamma CE_1(B_{\tau_\epsilon})}E_{\tau_{\epsilon i}}\left[dCE_1\right] + \frac{\gamma^2}{2}e^{-\gamma CE_1(B_{\tau_\epsilon})}E_{\tau_{\epsilon i}}\left[dCE_1^2\right]$$

$$= 0$$

Thus we have

$$E_{\tau_{\epsilon i}}\left[de^{-\gamma CE_1(B_{\tau_\epsilon})}\right] = \begin{cases} -e^{-\gamma CE_1}\frac{\tau_i}{2}\frac{d\tau_{\epsilon i}}{\tau_{v,i}^2} & \text{if } x_i^u \in (a(p), b(p)) \\ \\ 0 & \text{otherwise} \end{cases}$$

$$= -e^{-\gamma CE_1}\frac{\tau_i}{2}\frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}I(\text{uncostr})$$

For marginal value of information (MVI), we finally get

$$\text{MVI} = -\frac{\frac{dU_0}{d\tau_{\epsilon i}}}{\gamma U_0} = \frac{\tau_i}{2\tau_{v,i}^2}\frac{E\left[e^{-\gamma CE_1}I(\text{uncostr})\right]}{E\left[e^{-\gamma CE_1}\right]}.$$

b) In this case, the MVI is defined as

$$\text{MVI} = -\frac{d}{d\tau_{\epsilon i}}E[CE_1] = -\frac{d}{d\tau_{\epsilon i}}E[E_{\tau_{\epsilon i}}[CE_1]].$$

From the calculations above, we know

$$
E_{\tau_{\epsilon i}}\left[CE_1\right] = \begin{cases} \left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i}\frac{\gamma}{2}E_{\tau_{\epsilon i}}\left[(x_i^u)^2\right] + \frac{\tau_i}{2\gamma}\frac{d\tau_{\epsilon i}}{\tau_{v,i}^2} & \text{if } x_i^u \in (a(p), b(p)) \\[2mm] \frac{\gamma}{2}a(p)^2\left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i} & \text{if } x_i^u < a(p) \\[2mm] \frac{\gamma}{2}b(p)^2\left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i} & \text{if } x_i^u > b(p) \end{cases}
$$

$$
= \frac{\gamma}{2}E_{\tau_{\epsilon i}}[x_i^2]\left(\frac{1}{\tau_{v,i}}\right)^2 d\tau_{\epsilon i} + \frac{\tau_i}{2\gamma}\frac{d\tau_{\epsilon i}}{\tau_{v,i}^2}I(\text{unconstr})
$$

Thus,

$$
\text{MVI} = \frac{\gamma}{2\tau_{v,i}^2}E[x_i^2] + \frac{\tau_i}{2\gamma\tau_{v,i}^2}Pr(\text{unconstr})
$$

Assume that investor i's constraints become tighter and all other investors' constraints are not altered. This implies that prices are not affected. Consider the marginal value of information for investor i. In case 1, the only term affected by constraints is $\frac{U_0^u}{U_0}$. It decreases as investor i's constraints become tighter. This implies that marginal value of information decreases for investor i. We can use similar argument for case 2. ■

Proof. (Proposition 4) In case 1, we write the expression for the marginal value of information as

$$
\text{MVI} = \frac{\tau_i}{2\tau_{v,i}^2\gamma}\underbrace{\frac{U_0^u}{U_0}}_{\text{the term due to constraints}} \tag{20}
$$

In the case of risk-neutral market maker, constraints do not alter prices i.e., prices are independent of portfolio constraints. The only term affected by constraints is $\frac{U_0^u}{U_0}$. Consider first the nominator: $U_0^u = E[-e^{-\gamma CE_1}\mathbb{I}(x_i^u = x_i)]$. It increases (becomes less negative) as constraints become tighter: recall that investors get negative utility; as constraints become tighter, they get it in fewer states of the world. The denominator U_0 decreases (becomes more negative) as with constraints the certainty equivalent CE_1 in all states weakly decreases. Thus, the ratio decreases as constraints become tighter.

In case 2, the marginal value of information is given by

$$\text{MVI} = \frac{\gamma}{2\tau_{v,i}^2}\mathbb{E}[x_i^2] + \frac{\tau_i}{2\gamma\tau_{v,i}^2}\pi_2 \tag{21}$$

Again, prices are independent of portfolio constraints. As constraints become tighter, the first term $\mathbb{E}[x_i^2]$ decreases and the second term pi_2 also decreases. This implies that the marginal value of information decreases and hence the equilibrium acquisition decreases. Informational efficiency of prices beta still satisifes equation 3 and by proposition 1, β decreases as investors acquire less information. ∎

Lemma 1. *If x is normally distributed with mean μ and variance σ, then*

-
$$\int_l^m exp\left(-\frac{x^2}{2}\right) dF(x) = \frac{1}{\sqrt{1+\sigma^2}} exp\left(-\frac{1}{2}\frac{\mu^2}{1+\sigma^2}\right)\left(\Phi\left(\frac{m - \frac{\mu}{1+\sigma^2}}{\sqrt{\frac{\sigma^2}{1+\sigma^2}}}\right) - \Phi\left(\frac{l - \frac{\mu}{1+\sigma^2}}{\sqrt{\frac{\sigma^2}{1+\sigma^2}}}\right)\right)$$

-
$$\int_l^\infty exp\left(-kx\right) dF(x) = exp\left(\frac{k^2\sigma^2}{2} - \mu k\right)\Phi\left(\frac{\mu - l - k\sigma^2}{\sigma}\right)$$

Proof. (Proposition 6) Note that with risk-neutral market maker, the margins are given by

$$m^+(p) = m^-(p) = \frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}}$$

where

$$\frac{1}{\tau_m} = \frac{1}{\tau_v + \beta^2\tau_z} + \frac{1}{\tau_\theta}.$$

As informationally efficiency (β) decreases, margins increase. This implies that the constraint $a(p) = \frac{W_0}{m^+(p)}$ decreases and the constraint $b(p) = -\frac{W_0}{m^-(p)}$ increases. This implies that constraints tighten as informationally efficiency decreases. ∎

Appendix B: Application to Yuan (2005)

In this appendix, we apply our methodology developed in section 2 to study borrowing constraints introduced in Yuan (2005). In this case, Borrowing-constrained informed investor demand is bounded above by $b(p) = \delta_0 + \delta_1 p$ where $\delta_1 > 0$ and there is no lower bound on investor demand.

The borrowing constraint is a function of the price. The lower the asset price, the harder it is for informed investors to raise outside financing to invest in the risky asset.

$$f'(p) = \frac{c_p^m + (1 - \pi_3)c_p - \pi_3\delta_1}{(1 - \pi_3)c_\phi + c_\phi^m}$$

where all the coefficients are positive and π_3 denotes the mass of investors for which the constraint binds. The following theorem gives conditions under which there will be multiple equilibrium.

Proposition 7. *When the constraint is of the form* $b(p) = \delta_0 + \delta_1 p$ *where* $\delta_1 > 0$, *equilibrium is unique when* $\delta_1 < \frac{1}{\kappa_m}$ *and there will be multiple equilibria otherwise.*

Proof. (Proposition 7) In this case,

$$f'(p) = \frac{c_p^m + (1 - \pi_3)c_p - \pi_3\delta_1}{(1 - \pi_3)c_\phi + c_\phi^m} \tag{22}$$

where π_3 denotes the mass of investors for which the constraint binds. As p decreases, π_3 increases and numerator of equation (22) increases. In the extreme case, as p tends to low number, most of informed investors are binding and numerator tends to $c_p^m - \delta_1$. If this term is positive, we will always have unique equilibrium because $f'(p) > 0 \forall p$. If this term becomes negative, there could be multiple equilibria. ∎

What we mean by multiple equilibrium is that some realization of fundamentals can be supported by two prices. This results from the interaction of substitution and information effects

in the model. In typical REE model (Hellwig (1980b)), the substitution effect always dominates the information effect leading to unique equilibrium. In those models, the information effect is fixed as prices reveal the same amount of information regardless of level. In our setting, due to the borrowing constraint imposed on informed investors, unit change in price does not reflect the same information. This implies that information effect can dominate substitution effect for some realization of prices and there will be multiple equilibrium.

Appendix C: Equilibrium characterization with risk-averse market maker

With risk-averse market maker, the price can be written as $p = \mathbb{E}[v|p] - rp(p)$. We assume the financiers use information from prices to set margin in order to control VaR:

$$m^+(p) = \inf\{m^+(p) \geq 0 : Pr(p - v > m^+(p)|p) \leq 1 - \alpha\}.$$

$$m^-(p) = \inf\{m^-(p) \geq 0 : Pr(v - p > m^-(p)|p) \leq 1 - \alpha\}.$$

$m^+(p)$ and $m^-(p)$ are the margins on long and short positions (per unit of asset) respectively. We now derive the expressions for margins. To compute $m^+(p)$, we first determine the function $m_n^+(p)$ that satisfies

$$
\begin{aligned}
1 - \alpha &= Pr(E[v|p] - rp(p) - v > m_n^+(p)|p) \\
&= Pr\left(\sqrt{\tau_m}(E[v|p] - v) > \sqrt{\tau_m}(m_n^+(p) + rp(p))\big|p\right) \\
&= 1 - \Phi\left(\sqrt{\tau_m}(m_n^+(p) + rp(p))\right).
\end{aligned}
$$

Thus, we find

$$m^+(p) = [m_n^+(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}} - rp(p)\right]^+ \tag{23}$$

Similarly, one can define $m_n^-(p)$ which satisfies $Pr(v - p > m_n^-(p)|p) = 1 - \alpha$ and get

$$m^-(p) = [m_n^-(p)]^+ = \left[\frac{\Phi^{-1}(\alpha)}{\sqrt{\tau_m}} + rp(p)\right]^+ \tag{24}$$

The endogenous VaR margins are determined by three variables. Both margins on long and short positions increase in the exogenous level of confidence α and decrease in the endogenous informational efficiency of price β (through $\tau_m = \tau_v + \beta^2 \tau_z$). In addition, the margin on long (short) position decreases (increases) in the endogenous risk premium $rp(p)$. We would like to emphasize the fact that informational efficiency of price affects the tightness of margin constraint.

Formally, our financial market equilibrium with endogenous margin constraints is defined as follows: (1) financiers and investors determine demands and margins anticipating a particular price function (2) in equilibrium demands and margins are consistent with anticipated price function. We hold the precisions of investors' signals fixed.

Proposition 8. *(Equilibrium with endogenous margin requirements) When the portfolio constraints are of the form of margin as in equation (10) and margins are endogenously determined by VaR, there exists a unique generalized linear equilibrium. Moreover, in this unique equilibrium the function $f(p)$, i.e. the sufficient statistic ϕ, is increasing in price.*

Proof. (Proposition 8) One can prove that for every p there exists unique $\phi = f(p)$ such that market clears similarly to Proposition 2. We now prove that $f(p)$ is invertible. We plug expression for our endogenous margins into ODE (6) assuming that both m_n^+ and m_n^- are positive. We get

$$f'(p) = \frac{c_p^m + \pi_2 c_p - \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2}\right) rp(p)'}{\pi_2 c_\phi + c_\phi^m}.$$

from which, accounting for defn of $rp(p)$ we find

$$f'(p) = \frac{c_p^m + \pi_2 c_p + \kappa_m \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2} \right) c_p^m}{\pi_2 c_\phi + c_\phi^m + \kappa_m \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2} \right) c_\phi^m}.$$

Both m_n^+ and m_n^- are positive, when $f(p) \in \left[-\Phi^{-1}(\alpha) \frac{\sigma_{v|p}}{c_\phi^m} \frac{1}{\kappa_m} - \frac{c_0^m}{c_\phi^m} + \frac{c_p}{c_\phi^m} p; \Phi^{-1}(\alpha) \frac{\sigma_{v|p}}{c_\phi^m} \frac{1}{\kappa_m} - \frac{c_0^m}{c_\phi^m} + \frac{c_p}{c_\phi^m} p \right] = $
$[f^-(p); f^+(p)]$ Proceeding similarly, one can get

$$f'(p) = \begin{cases} \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m} \frac{\pi_3 W_0}{m^+(p)^2} c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m} \frac{\pi_1 W_0}{m^-(p)^2} c_\phi^m}, & \text{if } f < f^-(p), \\[2em] \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m} \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2} \right) c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m} \left(\frac{\pi_1 W_0}{m^-(p)^2} + \frac{\pi_3 W_0}{m^+(p)^2} \right) c_\phi^m}, & \text{if } f^-(p) < f < f^+(p), \\[2em] \frac{c_p^m + \pi_2 c_p + \frac{1}{\kappa_m} \frac{\pi_1 W_0}{m^-(p)^2} c_p^m}{\pi_2 c_\phi + c_\phi^m + \frac{1}{\kappa_m} \frac{\pi_1 W_0}{m^-(p)^2} c_\phi^m}, & \text{if } f > f^+(p). \end{cases}$$

Clearly, the derivative above is always positive, which means that the equilibrium function $f(p)$ is invertible. Thus, for each fundamental ϕ there exists a unique p clearing the market. The initial condition for the ODE above can be found by clearing the market for a particular price, e.g., price $p = 0$. ∎

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